First-Fit Coloring of Trees in Random Arrival Model

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Graph Coloring

Graph

Graph $G = (V, E)$ with $n = |V|$ vertices.

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Function $c : V \mapsto \mathbb{Z}^+$ with $c(v) \neq c(w)$ for every $\{v, w\} \in E$.

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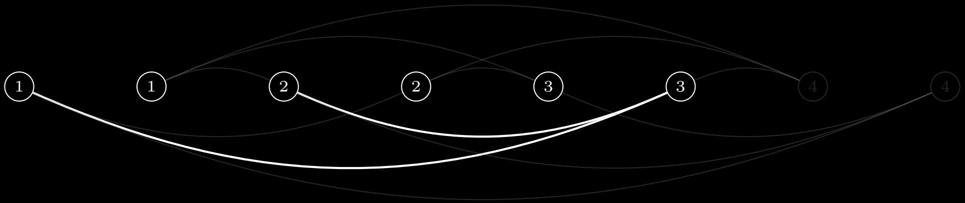
Chromatic number

$\chi(G)$ – minimum possible k with coloring $c : V \mapsto \{1, 2, \ldots, k\}$.

First-Fit Algorithm

Algorithm

1. Process vertices in order $v_1 \ll v_2 \ll \ldots \ll v_n$.
2. For vertex v_t use the first color that is not used as a color of any neighbor of v_j of v_t with $j < t$.



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First-Fit chromatic number

$\chi_{\mathsf{FF}}(G, \ll)$ – maximum color used by First-Fit on G in order $\ll$.

Question

Is First-Fit a good algorithm? How much does $\chi_{FF}(G, \ll)$ differ from $\chi(G)$?

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Answer

It all depends on the order $\ll$.

Optimistic Scenarios

Bipartite graph

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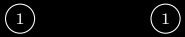
Bipartite graph



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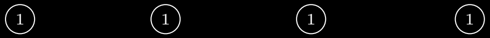

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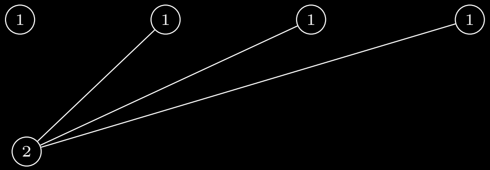

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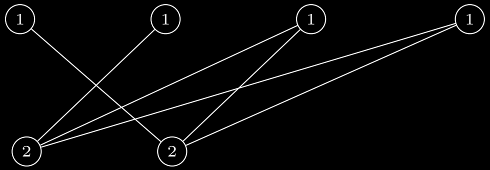

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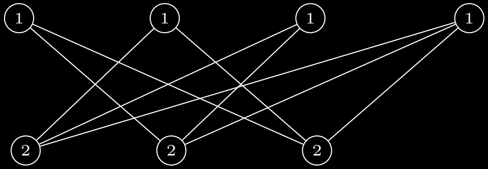

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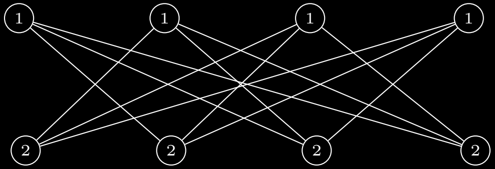

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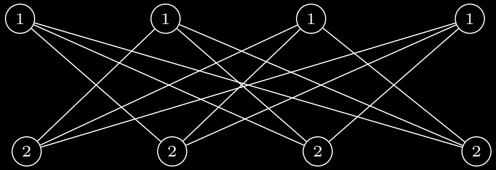

First-Fit can be optimal

1. Color G with k colors.
2. Construct $\ll$ by sorting vertices by their color.
3. First-Fit uses at most k colors in this order.

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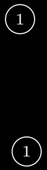
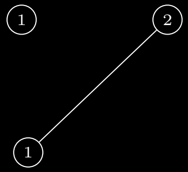


Pessimistic Scenarios

Bipartite graph



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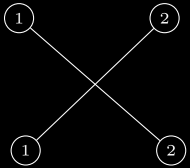

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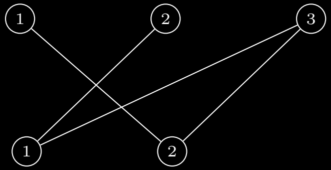

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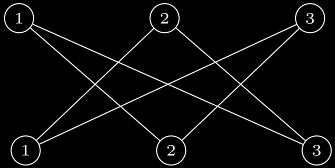

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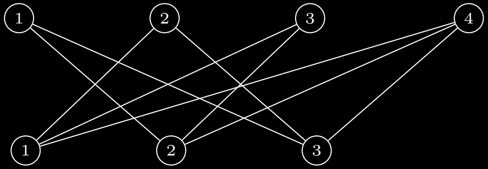

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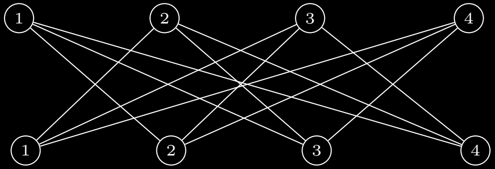

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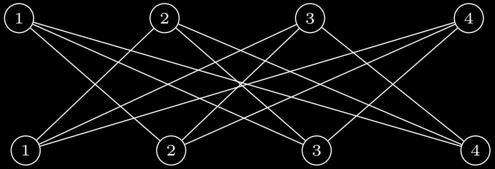

- $\chi(G) = 2$
- $\chi_{\mathsf{FF}}(G, \ll) = \frac{n}{2}$

Average Scenarios

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How many colors does First-Fit use on trees with n vertices, in expectation?

Interesting Settings

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Distributed Algorithms

Adding a random delay in distributed coloring algorithm allows to process vertices in random order.

First-Fit on Forests in Adversarial Order

Tree

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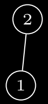

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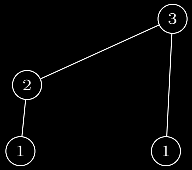

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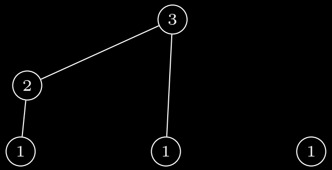

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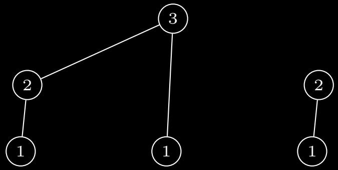

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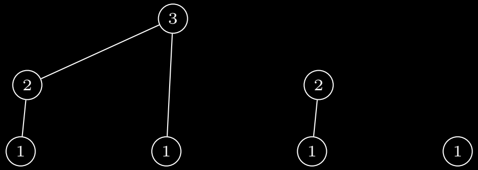

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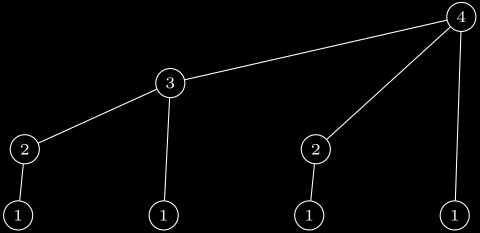

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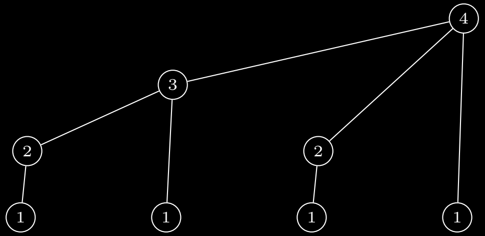

- $\chi(G) = 2$
- $\chi_{\mathsf{FF}}(G, \ll) = \log_2 n$

First-Fit on Forests in Random Order

Theorem

First-Fit uses $\left(\frac{1}{2} + o(1)\right) \cdot \frac{\ln n}{\ln \ln n}$ *colors on trees with* n *vertices.*

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Meaning

$\max_{G \text{ – tree with } n \text{ vertices}} \mathbb{E}_{\ll \text{ – random order of } V}\left[\chi_{\mathrm{FF}}(G, \ll)\right] \approx k$ for $n \approx k!$.

Upper bound

Proof (of a slightly weaker statement)

- $G^{\ll}$ is a directed graph with each edge in G directed towards the vertex later in $\ll$.

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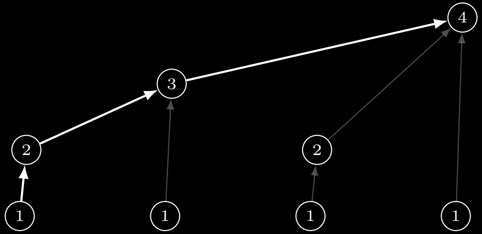

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- There are only n^2 different k-paths in the tree G.

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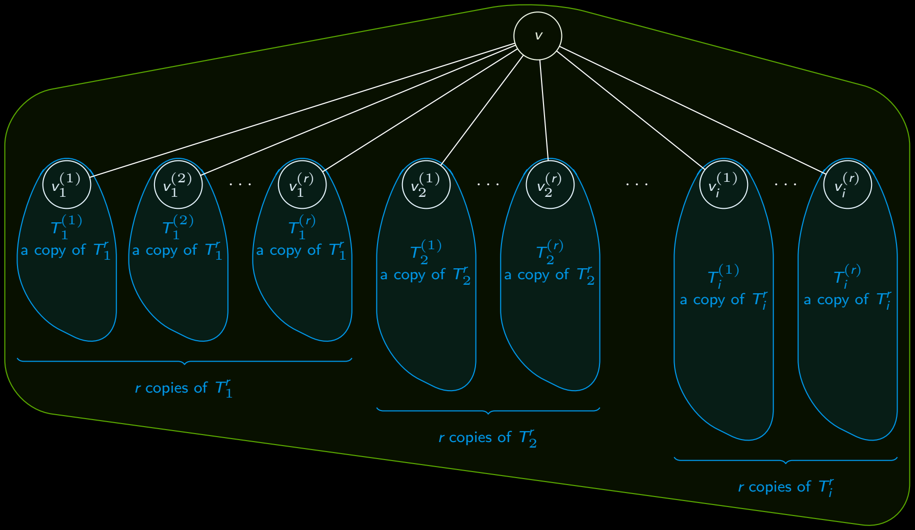
- Probability that a fixed one is directed in $G^{\ll}$ is $\approx \frac{1}{k!}$.
- For a constant probability of getting color k, we need $n^2 \approx k!$.

Lower bound



v
$v_1^{(1)}$
$T_1^{(1)}$
a copy of T_1^r
$v_1^{(2)}$
$T_1^{(2)}$
a copy of T_1^r
...
$v_1^{(r)}$
$T_1^{(r)}$
a copy of T_1^r
$v_2^{(1)}$
$T_2^{(1)}$
a copy of T_2^r
...
$v_2^{(r)}$
$T_2^{(r)}$
a copy of T_2^r
...
$v_i^{(1)}$
$T_i^{(1)}$
a copy of T_i^r
...
$v_i^{(r)}$
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a copy of T_i^r
r copies of T_1^r
r copies of T_2^r
r copies of T_i^r

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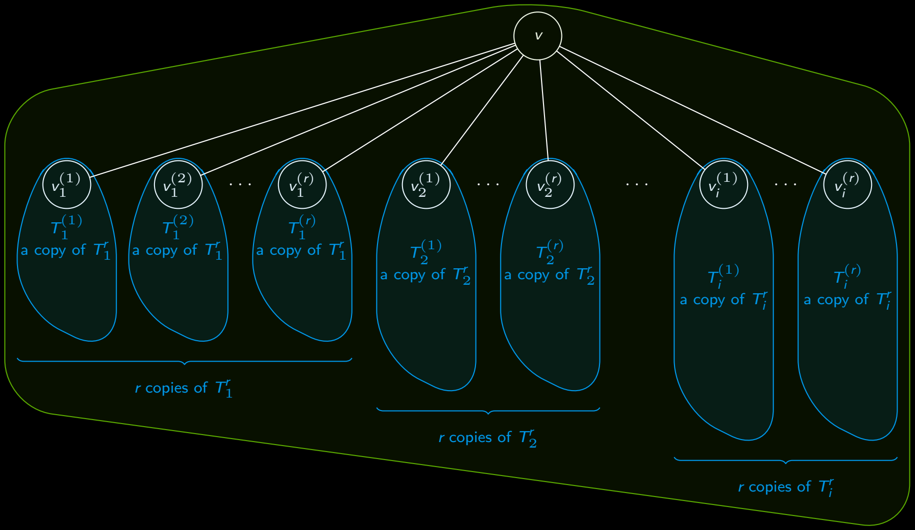
- T_1^r is a single vertex, T_{i+1}^r has r copies of each $T_1^r, \ldots, T_i^r$.

Lower bound



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$$\int_{\varepsilon_i}^{1} \left(1 - (1 - x + \varepsilon_i)^r\right)^i \partial x > 1 - \varepsilon_{i+1}$$

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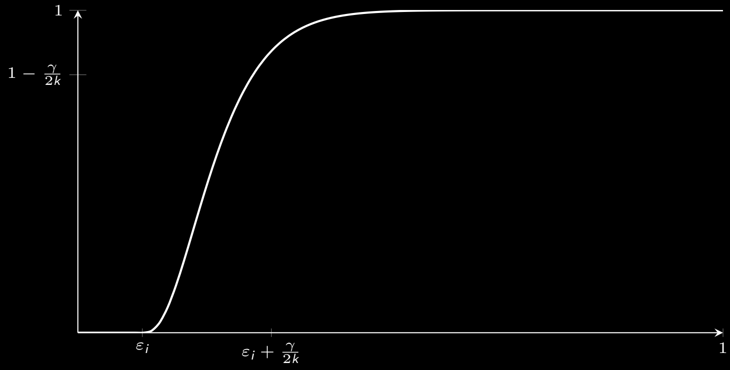

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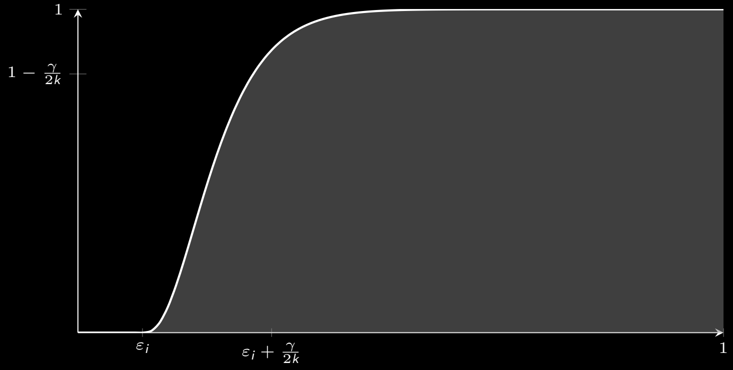

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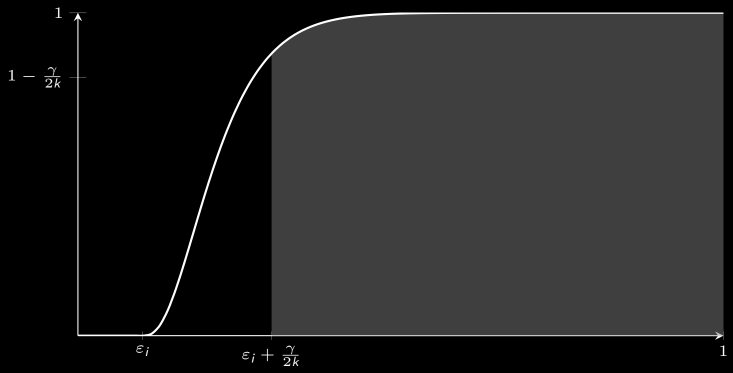

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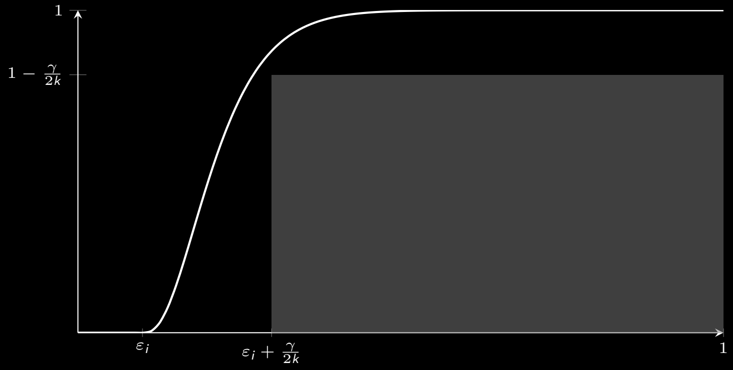

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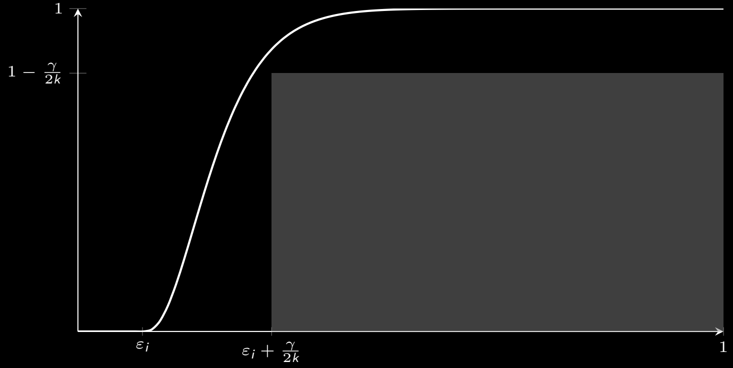
$$\left(1-\left(1-\frac{\gamma}{2k}\right)^r\right)^i > 1-\frac{\gamma}{2k}$$



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- $\int_{\varepsilon_i}^{1}(1 - (1 - x + \varepsilon_i)^r)^i \partial x > 1 - \varepsilon_{i+1}$
- T^r_k has $\approx k^k$ vertices.

Summary

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First-Fit uses $\left(\frac{1}{2}+o(1)\right)\cdot\frac{\ln n}{\ln\ln n}$ *colors on trees with* n *vertices.*

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Question

First-Fit on bipartite graphs

Thank You!